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SOLVING TIME-FRACTIONAL RADON DIFFUSION EQUATION USING CRANK-NICOLSON FINITE DIFFERENCE SCHEME

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ABSTRACT

This study introduces a finite difference numerical technique to simulate the solutions of the time-fractional Radon diffusion equation within a water medium. We develop the fractional order Crank-Nicolson finite difference scheme, utilizing time-fractional derivatives in the Caputo sense. We discuss the stability of the solution obtained by the developed Crank-Nicolson finite difference scheme. Furthermore, we delve into the convergence of the developed finite difference scheme. Lastly, we represent approximate solutions graphically with the help of Python programs.

Keywords: Time-Fractional Radon Diffusion Equation, Crank-Nicolson Finite Difference Method, Stability, Convergence Analysis, Python etc.

1 Introduction

Fractional calculus extends traditional calculus by accommodating non-integer or fractional orders of differentiation and integration, opening doors to modeling complex systems and phenomena. This field finds various applications across diverse domains such as Physics, Engineering, Signal Processing, Biology, Medicine, Chemical Engineering, Astronomy, and Astrophysics, among others [2, 3, 4, 6, 9, 17, 18]. Fractional partial differential equations describe intricate physical phenomena with memory and long-range dependencies, incorporating fractional order derivatives. Obtaining analytical solutions for such equations, especially in non-trivial cases, poses significant challenges due to the involvement of fractional derivatives. Finite difference methods, known for its versatility, are extensively employed for solving differential equations in multiple dimensions. They prove invaluable when analytical solutions are impractical or unavailable to study[1, 7, 8, 10, 11, 12, 15]. Radon, an odorless and colorless radioactive gas, occurs naturally through the radioactive decay of elements like uranium present in soil and rocks worldwide. It can migrate into the atmosphere and infiltrate both surface and underground water, posing health risks in both outdoor and indoor environments. Researchers extensively investigate its movement through various substances like soil, air, concrete, and activated charcoal [5, 13,14, 16]. Our aim is to study concentration of Radon in water medium.

In this paper, we consider the time-fractional order radon diffusion equation given below,

$$\frac{\partial^{\gamma} V(\zeta,\tau)}{\partial \tau^{\gamma}} = D\frac{\partial^{2} V(\zeta,\tau)}{\partial \zeta^{2}} - \lambda V(\zeta,\tau), 0 < \gamma \leq 1 \quad (1)$$

with initial condition:

$$V(\zeta,0) = 0, 0 < \zeta < L \quad (2)$$

and boundary conditions:

$$V(0,\tau) = V_0 \text{ and } \frac{\partial V(0,\tau)}{\partial \tau} = 0, \tau \geq 0 \quad (3)$$

where, λ is the decay constant, D is a diffusion coefficient, ζ and τ are spatial and temporal variables respectively. Time-fractional derivative in the above equation is considered in the Caputo sense, which is defined as follows:

Definition 1.1 *The Caputo time-fractional derivative of order* $\gamma, (0 < \gamma \leq 1)$ *is defined by,*

$$\frac{\partial^{\gamma} V(\zeta,\tau)}{\partial \tau^{\gamma}} = \frac{1}{\Gamma(1-\gamma)} \int_0^{\tau} \frac{\partial V(\zeta,\tau)}{\partial \eta} \frac{d\eta}{(\tau-\eta)^{\gamma}} \quad (4)$$

The subsequent sections of this paper are structured as follows: In section 2, we construct a Crank-Nicolson finite difference scheme tailored specifically for solving the time-fractional Radon diffusion equation. Section 3 delves into the scrutiny of the stability of the devised scheme, ensuring its robustness and reliability. Section 4 presents a rigorous proof of the convergence of our finite difference approximation, establishing its accuracy and efficacy. Finally, in the concluding section, we address a series of test problems, providing insightful illustrations of their solutions.

2 Finite Difference Scheme

Let $V(\zeta_i, \tau_k), i = 0,1,2,\ldots, M$ and $k = 0,1,2,\ldots, N$ be the exact solution of time fractional radon diffusion equation (1)-(3) at the mesh point (ζ_i, τ_k), where $\tau_k = k\tau,\ k = 0,1,2,\ldots,N$ and $\zeta_i = ih,\ i = 0,1,2,\ldots,M$, where $\tau = \frac{T}{N}$ and $h = \frac{L}{M}$. Let V_i^k be the numerical approximation of the point $V(ih, k\tau)$.

We approximate time-fractional derivative in the Caputo sense as follows,

$$\frac{\partial^{\gamma} V(\zeta_i,\tau_k)}{\partial \tau^{\gamma}} \approx \frac{1}{\Gamma(1-\gamma)} \sum \frac{V(\zeta_i,\tau_{j+1}) - V(\zeta_i,\tau_j)}{\tau} \int_{j\tau}^{(j+1)\tau} \frac{d\eta}{(\tau_{k+1}-\eta)} + O(\tau)$$

$$= \frac{1}{\Gamma(1-\gamma)} \sum_{j=0}^{k} \frac{V(\zeta_i,\tau_{j+1}) - V(\zeta_i,\tau_j)}{\tau} \int_{k-j)\tau}^{(k-j+1)\tau} \frac{d\xi}{\xi^{\gamma}} + O(\tau)$$

$$= \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} [V_i^{k+1} - V_i^k] + \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{j=1}^{k} b_j [V(\zeta_i, \tau_{k+1-j}) - V(\zeta_i, \tau_{k-j})] + O(\tau)$$

where $b_j = (j+1)^{1-\gamma} - j^{1-\gamma}, j = 0,1,2,\ldots,N$.

Now, we adopt the second order central difference scheme in space for each interior grid point $\zeta_i, 0 \leq i \leq M$. Therefore,

$$\frac{\partial^2 V(\zeta_i,\tau_k)}{\partial \tau^2} = \frac{1}{2}[\delta_x^2 u_i^{k+1} + \delta_x^2 u_i^k]$$

$$= \frac{1}{2}\left[\frac{u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}}{h^2} + \frac{u_{i-1}^{k} - 2u_i^{k} + u_{i+1}^{k}}{h^2}\right] \tag{5}$$

where, δ_x is the central difference operator. Using time fractional approximation the Crank-Nicolson type numerical approximation to equation (1)-(3) is given as follows,

$$\frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\left[V_i^{k+1} - V_i^k\right] + \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\sum_{j=1}^{k} b_j\left[V(\zeta_i, \tau_{k+1-j}) - V(\zeta_i, \tau_{k-j})\right]$$

$$= \frac{D}{2}\left[\frac{V_{i-1}^{k+1} - 2V_i^{k+1} + u_{i+1}^{k+1}}{h^2} + \frac{V_{i-1}^{k} - 2V_i^{k} + V_{i+1}^{k}}{h^2}\right] - \lambda V(\zeta_i, \tau_k)$$

where $b_j = (j+1)^{1-\gamma} - j^{1-\gamma}$. After simplification, we obtain

$$[V_i^{k+1} - V_i^k] + \sum_{j=1}^{k} b_j[V\zeta_i, \tau_{k+1-j}) - V(\zeta_i, \tau_{k-j})] = \frac{D\Gamma(2-\gamma)\tau^{-\gamma}}{2h^2}[V_{i-1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1} + V_{i+1}^{k} - 2V_i^{k} + V_{i+1}^{k}] - \lambda\Gamma(2-\gamma)\tau^{\gamma}C_i^k$$

Let $r = \frac{D\Gamma(2-\gamma)\tau^{-\gamma}}{2h^2}$ and $\mu = \lambda\Gamma(2-\gamma)\tau^{\gamma}$, we get

$$[V_i^{k+1} - V_i^k] + \sum_{j=1}^{k} b_j[V(\zeta_i, \tau_{k+1-j}) - V(\zeta_i, \tau_{k-j})]$$
$$= r[V_{i-1}^{k+1} - 2V_i^{k+1} + V_{i+1}^{k+1} + V_{i-1}^{k} - 2V_i^{k} + V_{i+1}^{k}] - \mu V_i^k$$

After simplification, we get

$$-rV_{i-1}^{k+1} + (1+2r)V_i^{k+1} - rV_{i+1}^{k+1} = rV_{i-1}^{k} + (1-2r-\mu)V_i^k + rV_{i+1}^k - \sum_{j=1}^{k} b_j[V_i^{k-j+1} - V_i^{k-j}] \tag{6}$$

Where $r = \frac{D\Gamma(2-\gamma)\tau^{-\gamma}}{2h^2}, \mu = \lambda\Gamma(2-\gamma)$ and $b_j = (j+1)^{1-\gamma} - j^{1-\gamma}$

From equation (6), we get

$$-rV_{i-1}^{k+1} + (1+2r)V_{i-1}^{k+1} - rV_{i+1}^{k+1}$$
$$= rV_{i-1}^{k} + (1-2r-\mu)V_i^k + rV_{i+1}^k - [b_1(V_i^k - V_i^{k-1}) + b_2(V_i^{k-1} - V_i^{k-2}) + b_3(V_i^{k-2} - V_i^{k-3}) + \cdots + b_{k-1}(V_1^2 - V_1^2) + b_k(V_i^1 - V_i^0)] - rV_{i-1}^{k+1} + (1 + 2r)V_i^{k+1} - rV_{i+1}^{k+1}$$

$$= rV_{i-1}^{k} + (1-2r-\mu)V_i^k + rV_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})V_i^{k-1} + b_kV_i^0$$

The initial condition is approximated as $V_i^0, i = 0,1,2, \dots, M$. For the two boundary points ζ_0 and ζ_M, the corresponding discretization schemes are, $V_0^k = V_0$ and $\frac{\partial V(1,\tau)}{\partial \zeta} = \zeta$ implies $V_{M+1}^k = V_{K-1}^k$.

Therefore, the fractional approximated initial boundary value problem is as follows:

For $k = 0$,

$$-rV_{i-1}^1 + (1+2r)V_i^1 - rV_{i+1}^1 = rV_{i-1}^0 + (1-2r-\mu)V_i^0 + rV_{i+1}^0 - rV_{i-1}^{k+1} + (1+2r)V_{i-1}^{k+1} - rV_{i+1}^{k+1} \quad (7)$$

For $k \geq 0$,

$$= rV_{i-1}^k + (1-2r-\mu-b_1)V_i^k + rV_{i+1}^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})V_i^{k-1} + b_k V_i^0 \quad (8)$$

The initial conditions,

$$V_i^0, i = 0,1,2, \dots, M. \quad (9)$$

The boundary conditions, $V_0^k = V_0$ and

$$V_{M+1}^k = V_{M-1}^k; k = 0,1,2, \dots, N. \quad (10)$$

where $r = \frac{\partial \Gamma(2-\gamma)\tau^{-\gamma}}{2h^2}, \mu = \lambda\Gamma(2-\gamma)\tau^\gamma$ and $b_j = (j+1)^{1-\gamma)} - j^{1-\gamma}, j = 1,2,3, \dots, k$.

Therefore, the fractional approximated initial boundary value problem (7)-(10) can be written in the following matrix equation form

$$AV^1 = BV^\circ + S \quad (11)$$

$$AV^{k+1} = CV^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})V^{k-1} + b_k V^0 + S \quad (12)$$

where

$$A = [a_{ij}] = \begin{cases} 1+2r & \text{if} i = j \\ -r & \text{if} i = j+1 \\ -r & \text{if} i = j-1 \\ -2r & \text{if} i = M, j = M-1 \\ 0 & \text{otherwise} \end{cases}$$

$$A = [a_{ij}] = \begin{cases} 1-2r-\mu & \text{if} i = j \\ r & \text{if} i = j+1 \\ r & \text{if} i = j-1 \\ 2r & \text{if} i = M, j = M-1 \\ 0 & \text{otherwise} \end{cases}$$

$$A=[a_{ij}]=\begin{cases}1-2r-\mu-b_1 & \text{if} i=j\\ r & \text{if} i=j+1\\ r & \text{if} i=j-1\\ 2r & \text{if} i=M, j=M-1\\ 0 & \text{otherwise}\end{cases}$$

$V^1=[V_1^1,V_2^1,V_3^1,\cdots,V_M^1]^t$, $V^k=[V_1^k,V_2^k,V_3^k,\cdots, V_M^k]^t$, $V^\circ=[V_1^0,V_2^0,V_3^0,\cdots,V_M^0]^t, S=[rV_0^1,0,0,\cdots,0]^t, r=\frac{D\Gamma(2-\gamma)\tau^{-\gamma}}{2h^2}, \mu=\lambda\Gamma(2-\gamma)\tau^\gamma, b_j=(j+1)^{1-\gamma}-j^{1-\gamma}, j=1,2,\cdots,k, i=0,1,\cdots,M, k=0,1,\cdots,N$. In the next subsection, we discuss the question of stability.

3 Stability

Lemma 3.1 *If $\lambda_j(A), j=1,2,3,\cdots,M$ represents the eigenvalues of matrix A,then*

1. $\lambda_j(A)\geq 1$
2. $\parallel A^{-1}\parallel_2\leq 1$
3. $\parallel B\parallel_2< 1$
4. $\parallel C\parallel_2< 1$

Proof: The Grschgorinâ€™s theorem states that each eigenvalues λ of matrix A is an at least one of the following disk.

$$|\lambda-a_{ij}|=\sum_{t=1,t\neq j}^{M} a_{ij}, j=1,2,3,\cdots,M$$

Therefore, the eigenvalue λ of the matrix A satisfies at least one of the following inequality,

$$|\lambda|\leq|\lambda-a_{ij}|+\sum_{t=1,t\neq j}^{M} a_{ij}\leq\sum_{t=1,t\neq j}^{M} a_{ij}; 1\leq j\leq M$$

and

$$|\lambda|\geq|a_{ij}|-|\lambda-a_{ij}|\geq|a_{ij}|-\sum_{t=1,i\neq j}^{M} a_{ij}; 1\leq j\leq M \tag{13}$$

Now, each eigenvalue λ of matrix A satisfy at least one of the following inequalities.

$$\lambda_j(A)\geq 1+2r-r=1+r>1;\ since\ r>0$$

Therefore,

$$\lambda_j(A)\geq 1$$

(ii) We have,

$$\parallel A\parallel_2=\max\zeta_{t\leq j\leq M-1}\parallel\lambda_j(A)|\geq 1$$

Hence,

$$\| A \|_2 \geq 1$$

Therefore,

$$\| A^{-1} \|_2 \leq \frac{1}{|\lambda_j(A)|} \leq 1$$

(iii) $$\| B \|_2 \leq r + 1 - 2r - \mu + r < 1$$

$$\| C \|_2 \leq |r + 1 - b_1 - 2r + r - \mu| \leq (1 - \mu) - b_1 \leq 1 - b_1 < 1$$

Theorem 3.2 *The solution of the finite difference scheme (7)-(10) for TFRDE (1)-(3) is unconditionally stable.*

Proof: Now to prove that the above finite difference scheme in unconditionally stable, that is to show that,

$$\| V^k \|_2 \leq \| V^\circ \|_2 ; k \geq 1$$

From equation (2.8), we have

$$AV^1 = BV^\circ, for\ k = 0$$

$$V^1 = A^{-1}BV^\circ$$

$$\| V^1 \|_2 \leq \| A^{-1}BV^\circ \|_2$$

$$\leq \| A^{-1} \|_2 \| B \|_2 \| V^\circ \|_2$$

$$\leq \| V^\circ \|_2$$

Therefore,

$$\| V^1 \|_2 \leq \| V^\circ \|_2$$

Therefore, the result is true for $k = 1$. Assume that the result is true for k, that is,

$$\| V^k \|_2 \leq \| V^\circ \|_2$$

Now to prove, the result is true for k+1, therefore, from equation (2.14), we have,

$$AV^{k+1} = CV^k \sum_{j=1}^{k-1} (b_j - b_{j+1})V^{k-j} + b_k V^\circ$$

$$V^{k+1} = A^{-1}CV^k + A^{-1} \sum_{j=1}^{k-1} (b_j - b_{j+1})V^{k-j} + b_k A^{-1}V^\circ$$

$$\| V^{k+1} \|_2 \leq \| A^{-1} \|_2 \| V \|_2 \| V^k \|_2 + \| A^{-1} \|_2 \sum_{j=1}^{k-1} (b_j - b_{j+1}) \| V^{k-1} \|_2$$

$$+ b_k \| A^{-1} \|_2 \| V^\circ \|_2$$

$$\leq |1-b_1| \parallel V^\circ \parallel_2 + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \parallel V^\circ \parallel_2 + b_k \parallel V^\circ \parallel_2$$

$$\leq \left(1-b_1+\sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k\right) \parallel V^\circ \parallel_2$$

$$\leq (1-b_1+(b_1-b_2)+(b_2-b_3)+\cdots$$

$$+(b_{k-1}-b_k)+b_k) \parallel V^\circ \parallel_2$$

Therefore,

$$\parallel V^{k+1} \parallel_2 \leq \parallel V^\circ \parallel_2$$

Hence by induction, the result is true for all k.

$$\parallel V^{k} \parallel_2 \leq \parallel V^\circ \parallel_2, \forall k$$

This shows that, the scheme is unconditionally stable.

4 Convergence

We introduce the another vector for,

$$\overline{V}_t^k = [V(\zeta_0,\tau_k),\cdots,V(\zeta_i,\tau_k),\cdots,V(\zeta_{M-1},\tau_k)]^t$$

which represents the exact solution at the time level τ_k whose size is M. Therefore from the above discretization scheme,

$$A\overline{V}^1 = B\overline{V}^\circ + t^1 \quad (14)$$

$$A\overline{V}^{k+1} = B\overline{V}^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})\overline{V}^{k-j} + b_k\overline{V}^\circ + \tau^{k+1};\ for\ k \geq 1 \quad (15)$$

where,

$$A = [a_{ij}] = \begin{cases} 1+2r & \text{if} i = j \\ -r & \text{if} i = j+1 \\ -r & \text{if} i = j-1 \\ -2r & \text{if} i = M, j = M-1 \\ 0 & \text{otherwise} \end{cases}$$

$$A = [a_{ij}] = \begin{cases} 1-2r-\mu & \text{if} i = j \\ r & \text{if} i = j+1 \\ r & \text{if} i = j-1 \\ 2r & \text{if} i = M, j = M-1 \\ 0 & \text{otherwise} \end{cases}$$

$$A = [a_{ij}] = \begin{cases} 1 - 2r - \mu - b_1 & \text{if} i = j \\ r & \text{if} i = j+1 \\ r & \text{if} i = j-1 \\ 2r & \text{if} i = M, j = M-1 \\ 0 & \text{otherwise} \end{cases}$$

and τ^k is the vector of the truncation error at the time level t^k.

Theorem 4.1 *The finite difference scheme (7) - (10) for TFRDE (1) - (3) is unconditionally convergent, that is to prove* $\| E^k \|_2 \leq \| E^\circ \|_2, as\ (h, \tau) \to (0,0)$.

Proof. We subtract equation (7) from (14) and (10) from (15) respectively, we get

$$A(\overline{V}^1 - V^1) = B(\overline{V}^1 - V^\circ) + \tau^1 \quad (16)$$

$$A(\overline{V}^{k+1} - V^{k+1}) = B(\overline{V}^k - V^k) + \sum_{j=1}^{k-1} (b_j - b_{j+1})(\overline{V}^{k-1} - V^{k-j} + b_k(\overline{V}^\circ - V^\circ) + \tau^{k+1}$$
(17)

We set,

$$E^k = (\overline{V}^k - V^k)$$

where,

$$E^k = [e_1^k, e_2^k, e_3^k, \cdots, e_{m-1}^k]^\tau; C^k = [V_1^k, V_2^k, V_3^k, \cdots, V_{M-1}^k]^\tau$$

From equation (4.3), we have

$$AE^1 = BE^\circ + \tau^1$$

$$E^1 = A^{-1}BE^\circ + A^{-1}\tau^1$$

$$\| E^1 \|_2 \leq \| A^{-1} \|_2 \| B \|_2 \| E^\circ \| + \| A^{-1} \|_2 \| \tau \|^1 \|_2 \leq \| E^\circ \|_2 +$$

$$+O(\tau^{2-\gamma} + h^2)$$

Assume that the result is true for k,

$$\| E^k \|_2 \leq \| E^\circ \|_2 + 0(\tau^{2-\gamma} + h^2)$$

Now to prove that the result is true for $k + 1$. From equation (4.4), we have

$$AE^{k+1} = CE^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})E^{k-j} + b_k E^\circ + \tau^{k+1}$$

$$E^{k+1} = A^{-1}CE^{k} + A^{-1}\sum_{j=1}^{k-1}(b_j - b_{j+1})E^{k-j} + b_k A^{-1}E^{\circ}$$

$$+A^{-1}\tau^{k+1}$$

$$\| E^{k+1} \|_2 = \| A^{-1} \|_2 \| C \|_2 \| E^{k} \|_2 +$$

$$\| A^{-1} \|_2 \sum_{j=1}^{k-1}(b_j - b_{j+1}) \| E^{k-j} \|_2 + b_k \| A^{-1} \|_2 \| E^{\circ} \|_2$$

$$+\| A^{-1} \|_2 \, \tau^{k+1}$$

$$\leq |1 - b_1| \| E^{\circ} \|_2 + \sum_{j=1}^{k-1}(b_j - b_{j+1}) \| E^{\circ} \|_2 + b_k \| E^{\circ} \|_2$$

$$\leq \left(1 - b_1 + \sum_{j=1}^{k-1}(b_j - b_{j+1})b_k\right) \| E^{\circ} \|_2 + 0(\tau^{2-\gamma} + h^2)$$

$$\leq \| E^{\circ} \|_2 \, 0(\tau^{2-\gamma} + h^2)$$

Thus the result is true for $k+1$, hence by induction it is true for all k.

$$\| E^{k} \|_2 \leq \| E^{\circ} \|_2, as\ (h,\tau) \rightarrow (0,0)$$

This proves that, the scheme is unconditionally convergent.

5 Numerical Solution

We consider the following time-fractional radon diffusion equation

$$\frac{\partial^{\gamma} V(\zeta,\tau)}{\partial \tau^{\gamma}} = D\frac{\partial^{2} V(\zeta,\tau)}{\partial z^{2}} - \lambda V(\zeta,\tau)$$

Initial condition: $V(\zeta,0) = 0, 0 < z < L$

Boundary conditions: $V(0,\tau) = dcV_0,$ and $\frac{\partial V(L,\tau)}{\partial \tau} = 0, \tau \geq 0.$

Exact solution for $\gamma = 1$ is as follows,

$$V(\zeta,\tau) = dcV_0\left[\frac{\cosh\sqrt{\frac{\mu}{D}}(L-\zeta)}{\cosh\sqrt{\frac{\mu}{D}}L} - \frac{D\pi}{L^2}\sum_{n=0}^{\infty}\frac{(2n+1)e^{-\left(\frac{(2n+1)^2\pi^2 D}{4L^2}+\mu\right)\tau}}{\left(\frac{(2n+1)^2\pi^2 D}{4L^2}+\mu\right)}\sin\frac{(2n+1)\pi\zeta}{2L}\right] \tag{18}$$

In Table 1, the approximate solution of time-fractional radon diffusion equation derived from the fractional finite difference scheme is compared with the exact solution for the parameters $\mu = 2.1 \times 10^{-6}, T = 1,\ L = 1, D = 1 \times 10^{-9}, V(0,\tau) = 1,$ which demonstrating the method's effectiveness.

	Absolute error				
$x \rightarrow$	0.2	0.4	0.6	0.8	1.0
$t \downarrow$					
.0	1.03×10^{-4}	5.41×10^{-5}	3.93×10^{-5}	3.34×10^{-5}	3.18×10^{-5}
.2	8.45×10^{-5}	4.44×10^{-5}	3.22×10^{-5}	2.74×10^{-5}	2.61×10^{-5}
.4	6.94×10^{-5}	3.64×10^{-5}	2.65×10^{-5}	2.25×10^{-5}	2.14×10^{-5}
.6	5.69×10^{-5}	2.99×10^{-5}	2.17×10^{-5}	1.85×10^{-5}	1.76×10^{-5}
.8	4.67×10^{-5}	2.45×10^{-5}	1.78×10^{-5}	1.51×10^{-5}	1.44×10^{-5}
.0	3.83×10^{-5}	2.01×10^{-5}	1.46×10^{-5}	1.24×10^{-5}	1.18×10^{-5}

Table 1: Absolute error

Now, we take another set of particular values for the parameters as follows: Radon diffusivity coefficient in water $D = 1 \times 10^{-9} Bq/m^3$, Spatial length $L = 1.7278\,cm$, Radon decay constant $\mu = 2.1 \times 10^{-6}$, Adsorption coefficient $c = 4m^2/kg$, Material density $d = 0.5g/cm^3$ and Constant Radon concentration in air $V_0 = 200Bq/m^3$. With this parameters, we simulate the radon concentration in water after $\tau = 5.6\ min$ in the Figure 1 and we can observe that radon loss in its concentration with length.

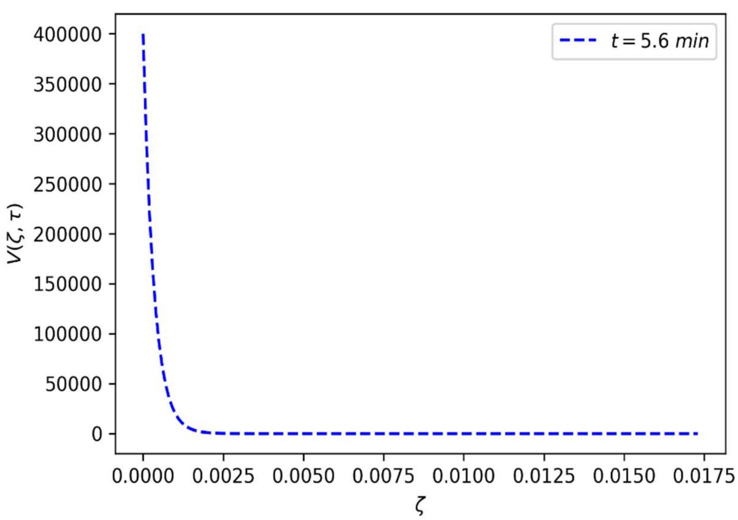

Figure 1: Radon concentration for $\tau = 12\ hrs, \gamma = 1, h = 0.001, \tau = 432$

Also, in Figure 2, we present radon concentration for $\gamma = 1$ and $\gamma = 0.99$ and observe that radon concentration increases more rapidly as γ increases.

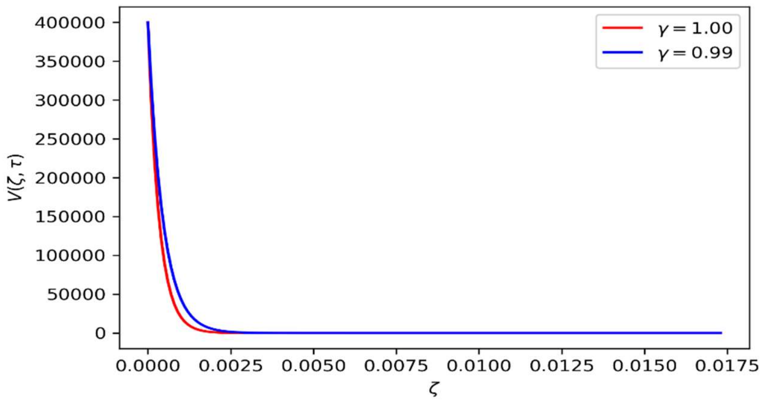

Figure 2: Radon concentration for $t = 12\ \ hrs, h = 0.001, \tau = 432$

6 Conclusion

i) We have successfully developed Crank-Nikolson finite difference scheme for solving time-fractional order Radon diffusion equations.

ii) A comprehensive analysis of the stability and convergence of the proposed scheme has been conducted.

iii) Utilizing this method, numerical solutions for practical problems involving a water medium have been obtained, and these solutions have been presented through graphical simulations.

iv) The effect of the time-fractional order, denoted by γ, on radon concentration has been investigated, revealing a rapid decrease in concentration as γ increases.

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