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MATHEMATICAL CHARACTERIZATION OF A NEW SIZE BIASED FAILURE DISTRIBUTION

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Abstract
The present study offers a mathematical study of a new size biased distributions which has been developed for a failure distribution. Different important properties of the new distribution have been obtained which includes mgf, CV, skewness, kurtosis and the distribution of order statistics.. The estimation of its parameter has been discussed using different methods.

Keywords : Size-biased distributions, Failure Distribution, Order statistics, maximum-likelihood estimation.

1.Introduction

The statistical analysis of any data strongly depends on the assumed probability distribution from where the data are being taken. Because of this, the considerable efforts have been put in the development of large number of new probability distributions so that the use of approximation could be minimized. In many cases we are forced to use a certain distribution while we know that the data are not exactly following that distribution but approximately following that distribution. However, there are still many important problems where the actual data does not follow any of the standard probability models.

Size-biased distributions are the special cases of the weighted distributions and arise in practice when observations from a sample are recorded with probability proportional to some measure of
unit size and provide a unifying approach for the problems where the observations fall in the non-experimental, non-replicate and non-random categories.

If the random variable X has the probability density function (pdf), *f (x; θ)* then the size-biased distribution takes the following form,

$$f(x;\theta) = \frac{x^a f(x)}{Mean}$$

for α = 1, and α = 2 we get the size-biased and area-biased distributions and is applicable for area-biased sampling. In many statistical sampling situations care must be taken so that one does not inadvertently sample from size-biased distribution in place of the one intended. Fisher (1934) firstly introduced these distributions to model ascertainment biases which were later formalized by Rao (1965) in a unifying theory. Size-biased observations occur in many research areas and its fields of applications includes econometrics, environmental science, medical science, sociology, psychology, ecology, geological sciences etc. The applications of size-biased distribution theory has been used by many research scholars Patil and Ord (1977s) studied the size-biased sampling and the related form-invariant weighted distribution. Much research have been done relating to size-biased distributions and their applications in different areas of knowledge by many researchers including Patil and Rao (1978), Gove (2003), Das and Roy (2011), Ducey and Gove (2015), Bashir and Rasul, (2015) and Ayesha (2017), are some among others [11 - 16]. The new failure model probability density function

$$f(x) = \theta^{-x}\log\theta \qquad (1)$$

In this work, we introduced a new distribution called size biased New failure Model (SBZD), where the density function is given:

$$f(x) = \begin{cases} x\theta^{-x}(\log\theta)^2 & ,x,\theta > 0 \\ 0 & ,otherwise \end{cases} \qquad (2)$$

The SBNFM is motivated by the following: the SBZ distribution use may be restricted to the tail of a distribution, but it is easy to apply. The formulas of the mean, variance, mean deviation, entropy and the quantile function are simple in form and may be used as quick approximations in many cases.

The main advantage of using sized-based distributions appears when the sample is recorded with unequal probabilities. Accordingly, the superiority of the SBNFM is illustrated to ball bearings data. It is shown that the SBNFM is the most appropriate model for this data set as

compared to others distributions. We believe that the SBNFM distribution is an alternative distribution to lifetime data analysis.

The paper is organized as follows. In Section 2, we introduce the SBNFM, and give immediate properties as the mode, cumulative, survival and hazard rate functions, plots of the density and cumulative functions for some parameter values. Section 3 deals on the moments, and extreme order statistics. In Section 4, we are interested in parameters estimation the maximum likelihood estimation In this last section, Mean residual function, and Stress Strength Parameter.. We finish the paper with a concluding remarks.

Statistical and Reliability Measures of Size Biased New Failure Model.(SBNFM) In this section, we give the size biased New Failure model on and study its properties. Let X be random variable with PDF and CDF :

$$f(x) = \begin{cases} x\theta^{-x}(\log\theta)^2 & , x, \theta > 0 \\ 0 & , otherwise \end{cases}$$

And the CDF is

$$F(x) = 1 - \theta^{-x}[1 + x\log\theta] \tag{3}$$

And the first derivative of

$$\frac{d}{dx} f(x) = \frac{d}{dx} x\theta^{-x}(\log\theta)^2 = 0$$

Therefore, the mode of SBZGD is given by

$$Mode(x) = \begin{cases} \frac{1}{\log\theta} \\ 0 \end{cases}$$

2.1 Survival and hazard rate functions

The survival and failure rate (hazard rate) functions for a continuous distribution are defined as:

the Survival Function is

$$S(x) = \theta^{-x}[1 + x\log\theta] \tag{4}$$

the Hazard rate function is

$$H(x) = \frac{(x\log\theta)^2}{1+x\mathrm{lo}} \tag{5}$$

2.2 Moments and related measures

The rth moment about the origin of the Size Baised New failure model can be obtained as

$$\mu_r' = E(x^r) = \int_0^\theta x^r\, f(x)dx = \int_0^\theta x^r\, x\theta^{-x}(\log\theta)^2 dx$$

Using integration by parts, we get finally a general expression for the rth factorial moment of SBNFM as :

$$\mu_r' = \frac{(r+1)!}{(\log\theta)^r} \tag{6}$$

Substituting r = 1, 2, 3 and 4 in (..), the first four moments can be obtained and then using the relationship between moments about origin and moment about mean, the first four moment about origin of SBNFM were obtained as:

$$\mu_1' = E(x) = \int_0^\theta x\, f(x)dx = \frac{2}{\log\theta}$$

$$\mu_2' = E(x^2) = \int_0^\theta x^2\, f(x)dx = \frac{6}{(\log\theta)^2}$$

$$\mu_3' = E(x^3) = \int_0^\theta x^3 f(x)dx = \frac{24}{(\log\theta)^3}$$

$$\mu_4' = E(x^4) = \int_0^\theta x^4 f(x)dx = \frac{120}{(\log\theta)^4}$$

2.3 Moment generating function

In this sub section we derived the moment generating function of SBNFM. We begin with the well known definition of the moment generating function given by

MGF

$$M_x(t) = Ee^{tx} = \int_0^\theta e^{tx} f(x)dx$$

$$M_x(t) = \int_0^\theta e^{tx} x\theta^{-x}(\log\theta)^2 dx$$

$$M_x(t) = (\log\theta)^2 \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\theta x^{r+1} \theta^{-x} dx \qquad (7)$$

Characteristics Function

$$\phi_x(t) = Ee^{itx} = \int_0^\theta e^{itx} f(x)dx$$

$$\phi_x(t) = \int_0^\theta e^{itx} x\theta^{-x}(\log\theta)^2 dx$$

$$\phi_x(t) = (\log\theta)^2 \sum_{r=0}^{\infty} \frac{it^r}{r!} \int_0^\theta x^{r+1} \theta^{-x} dx \qquad (8)$$

2.4Order Statistics

We know that if $X_{(1)}, X_{(2)}X_{(3)} \ldots \ldots \ldots \ldots \ldots X_{(n)}$denotes the order statistics of a random sample $X_1, X_2X_3 \ldots \ldots \ldots \ldots \ldots X_n$from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$ then the pdf of r^{th} order
statistics is given by

$$f_{r;n}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)$$

The r^{th} order statistics for the SBNFM is obtained as;

$$f_{r;n}(x) = \frac{n!\,(\log\theta)^2}{(r-1)!\,(n-r)!}\sum_{i=0}^{n}\binom{n-r}{i}(-1)^i(1+x\log\theta)^{r+i-1}x\theta^{-x(r+i)}$$

Therefore, the pdf of the 1^{st} order statistics of SBNFM is is given by

$$f_{i;n}(x) = n(\log\theta)^2(1+x\log\theta)^{n-1}x\theta^{-x(n+2)} \tag{9}$$

Therefore, the pdf of the n^{th} order statistics of SBNFM is given by

$$f_{n;n}(x) = n(\log\theta)^2 x\sum_{i=0}^{n}\binom{n-1}{i}(-1)^i(1+x\log\theta)^i\theta^{-x(i+1)} \tag{10}$$

3.1Maximum Likelihood estimator

Maximum likelihood estimation has been the most widely used method for estimating the parameters of the size biased of new failure model. Let distribution. Let $x_1, x_2, x_3 \dots\dots\dots\dots\dots . x_n$ be a random sample from the size biased New Failure Model, then the corresponding likelihood function is given as

$$f(x) = x\theta^{-x}(\log\theta)^2$$

$$L = \prod_{i=0}^{n} x\,\theta^{-x}(\log\theta)^2$$

$$L = (\log\theta)^{2n}\theta^{-\sum x}\prod_{i=0}^{n} x$$

The log likelihood function is given as,

$$\log L = 2n\log(\log\theta) - \sum x\log\theta + \sum\log x$$

$$\hat{\theta} = \theta_{MLE} = e^{\frac{2}{\bar{x}}} \tag{11}$$

3.2Fisher Information

$$f(x) = x\theta^{-x}(\log\theta)^2$$

$$L = \prod_{i=0}^{n} x\,\theta^{-x}(\log\theta)^2$$

$$L = (\log\theta)^{2n}\theta^{-\sum x}\prod_{i=0}^{n} x$$

$$\log L = 2n\log(\log\theta) - \sum x\log\theta + \sum\log x$$

$$\frac{\partial}{\partial x}\log L = \frac{2n}{\theta\log\theta} - \frac{\sum x}{\theta}$$

$$\frac{\partial^2}{\partial x^2}\log L = \frac{\sum x}{\theta^2} - \frac{2n}{(\theta\log\theta)^2}(1+\log\theta)$$

3.3 Mean residual life function

Assuming that X is a continuous random variable with survival function, the mean residual life function is defined as the expected additional lifetime that a component has survived until time t. The mean residual life function, say, $\mu(t)$ is given by

$$\mu(t) = \frac{1}{P(X>t)}\int_t^{\infty} P(X>x)\,dx \ , t \geq 0$$

$$\mu(t) = \frac{1}{S(t)}\left\{E(t) - \int_0^t xf(x)\,dx\right\} - t, t \geq 0$$

Where

$$E(t) = \frac{2}{\log\theta}$$

$$\int_0^t xf(x)\,dx = \frac{2}{\log\theta} - t^2\theta^{-t} - \frac{2t\theta^{-t}}{\log\theta} - \frac{2\theta^{-t}}{\log\theta}$$

Finally

$$\mu(t) = \frac{t^2+2t+2}{1+t\log\theta} \qquad (12)$$

3.4 Stress-strength parameter:

If X_1 and X_2 are two continuous and independent random variables, where $X_1 \sim SBNFM(\theta_1)$ and $X_2 \sim SBNFM(\theta_2)$ then the stress strength parameter, say S, is defined as

$$S = \int_{-\infty}^{+\infty} f_1(x)\,F_2(x)dx$$

using the pdf and cdf of APP distribution, stress strength parameter S, can be obtained as

$$S = \log\theta_1 \int_0^{+\infty} x\theta_1^{-x}\,[1-\theta_1^{-x}(1+\log\theta_2)]dx$$

Finallay

$$S = \sum_{k_1=0}^{\infty}\frac{(-\log\theta_1)^{k_1+1}}{k_1!}\left[\sum_{k_1=0}^{\infty}\frac{(-\log\theta_2)^{k_2}}{k_2!}\left\{\frac{1}{k_1+k_2+2} - \frac{\log\theta_2}{k_1+k_2+3}\right\} - \frac{1}{(k_1+1)!}\right]$$

(13)

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